Kinnim (3:2): An Intuitive Mathematical Approach

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Introduction to *Kinnim*:

The *mishnayot* of *Kinnim* discuss bird sacrifices that have become intermingled. Bird sacrifices are either *olot* or *ḥatta'ot,* each following a different sacrificial procedure.[1] While there are instances of bird sacrifices with an obligation to bring only one or more *olot* to fulfill a *neder* or a *nedavah* or even to sacrifice one or more *ḥatta'ot,*[2] bird sacrifices are more commonly brought in pairs, as will be assumed throughout this paper. A *ḥovah*, an obligation, refers to a *ken* (a nest of birds) that consists of an **even** number of birds, half of whom must be sacrificed as *ḥatta'ot* and half as *olot.*

If a woman[3] were to obligate herself to bring a *ken* of say 8 birds, she may designate each of two groups of 4 birds as *ḥatta'ot* and *olot* respectively forming what is called a *ken mefureshet*, a designated nest; the Kohen would then sacrifice them accordingly. Alternatively, she might give all eight birds to the Kohen, in what is called a *ken stumah*, an undesignated nest, and the Kohen can sacrifice each bird as he chooses as long as 4 are sacrificed as *olot* and the other 4 as *ḥatta'ot.*

Most of the *mishnayot* in *Kinnim* deal with (an arbitrary number of) *kinnim*, either *stumot* or *mefurashot*, that become intermingled and discuss two sets of rules for *kinnim* that have become intermingled, depending on whether or not consultation preceded the Kohen's sacrifice. The language of the *Mishnah* at the beginning of the third chapter, introduces the chapter with "*ba-meh devarim amurim, be-Kohen nimlaḥ.*" As normally

[1] Where the blood of the *korban* is sprinkled, above or below a line around the middle of the *mizbayaḥ* called the *ḥut ha-sikrah*, differentiates the sacrifice of *ḥatta'ot* and *olot.*

[2] This can occur if a *ḥovah* was partially sacrificed.

[3] It is normally women who are required to bring these sacrifices after childbirth.

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understood by almost all commentators, this introductory phrase states that while the first two chapters only covered cases of prior consultation, this third chapter covers cases where there was no prior consultation.

The guiding principle where there is consultation is:

- Disallow any potentially incorrect sacrifice.

The guiding principle where there is no consultation is simple to state but on occasion difficult to compute:[4]

- Construct the worst case, invalidating as many birds as possible; the remaining number of birds is considered to have been sacrificed correctly.

Note that after the fact, we do not penalize a sacrifice done without consultation to the same number of valid birds as can be sacrificed under the *ab initio* rule. Rather, after the fact, a greater number of birds are often credited as having been sacrificed effectively.

Where both nests are of equal size, say each with 4 birds, then with consultation 4 of the 8 birds could be sacrificed, 2 birds as *olot* and 2 birds as *ḥatta'ot*. Regardless of whose birds are chosen, 2 birds may always be sacrificed as *olot* and 2 birds as *ḥatta'ot*. None of the 4 birds is sacrificed incorrectly.[5] In general, ½ of the intermingled nest can be sacrificed, ¼ as *olot* and ¼ as *ḥatta'ot*. Interestingly, as pointed out in the first mishnah in the third chapter, in a case of equal sized nests that were intermingled, even proceeding without consultation would not change that result. If all the birds in two equal sized nests that were intermingled are sacrificed, it is possible that all the birds in each of the two original nests are sacrificed identically, invalidating half of the birds. Thus, in the case of equal sized nests, with or without consultation, half of the combined nest will always be valid.

However, when 2 unequal sized nests are intermingled, proceeding without prior consultation gives a different result. For example, in the case of two nests with 2 and 4 birds respectively, the Kohen, assuming that he is dealing with a normal *ken stumah* of 6 birds, would proceed to

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4 I know of no instance where this rule is disputed. In those (very few) cases where a commentator appears to disagree, I believe it is more than likely that the *halakhic* rule is accepted but incorrectly computed in that (isolated) situation.

5 Note that a third bird cannot be sacrificed as an *olah*. Were that to be allowed, all three *olot* could have potentially come from the same nest where at most two *olot* may be sacrificed.

sacrifice 3 birds as *ḥatta'ot* and 3 birds as *olot*. Based on the accepted *halakhic* principle, the worst case must now be determined. It is easy to see that the worst case occurs if both birds of the smaller nest are sacrificed identically.[6] In that case, 3 of the birds from the larger nest are also sacrificed identically.[7] Now let us compute what was correctly sacrificed. From the smaller nest, 1 bird was sacrificed correctly (and 1 was not.) In the larger nest 1 pair of birds (an *olah* and a *ḥattat*) as well as 1 of the other 2 birds was sacrificed correctly; as a result, 3 birds from the larger nest were sacrificed correctly. In total, 4 of the 6 birds (1 from the smaller nest and 3 from the larger nest) were sacrificed correctly.

This case of unequal sized nests that are intermingled generalizes easily. If 2 *kinnim stumot* of 2*N and 2*M birds are intermingled where N<=M, then when there is consultation only 2*N birds can be sacrificed. However, without consultation, all the birds are sacrificed, and the owners get credit for 2*M birds. This rule, which I believe is accepted without dispute, is stated explicitly by Rambam.[8] This example is a special case of the term *merubah*, which we will now address more generally.

***Kinnim* (3:2) and the term *merubah*:**

Kinnim (3:2) reads as follows:[9]

> אַחַת לָזוֹ, וּשְׁתַּיִם לָזוֹ, וְשָׁלֹשׁ לָזוֹ, וְעֶשֶׂר לָזוֹ, וּמֵאָה לָזוֹ, עָשָׂה כֻלָּן לְמַעְלָה, מֶחֱצָה כָשֵׁר וּמֶחֱצָה פָסוּל. כֻּלָּן לְמַטָּן, מֶחֱצָה כָשֵׁר וּמֶחֱצָה פָסוּל. חֶצְיָן לְמַעְלָן וְחֶצְיָן לְמַטָּן, הַמְרֻבֶּה כָשֵׁר. זֶה הַכְּלָל, כָּל מָקוֹם שֶׁאַתָּה יָכוֹל לַחֲלֹק אֶת הַקִּנִּין וְלֹא יְהוּ מִשֶּׁל אִשָּׁה אַחַת, בֵּין מִלְמַעְלָן בֵּין מִלְּמַטָּן, מֶחֱצָה כָשֵׁר וּמֶחֱצָה פָסוּל. כָּל מָקוֹם שֶׁאֵין אַתָּה יָכוֹל לַחֲלֹק אֶת הַקִּנִּין עַד שֶׁיְּהוּ מִשֶּׁל אִשָּׁה אַחַת, בֵּין מִלְמַעְלָן בֵּין מִלְּמַטָּן, הַמְרֻבֶּה כָשֵׁר.

If one [pair] belonged to one woman and two [pairs] to another, or [even] three [pairs] to another, or [ten] pairs to another or a hundred to another, and he offered all of them above, then half are valid and half are invalid. [Similarly], if he offered all of them below, half are valid, and half are invalid. [If he offered] half of them above and half below, then the [number of birds as there is in the] larger part are valid. This is the

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6 If they were not sacrificed identically but instead one was sacrificed as an *olah* and the other as a *ḥattat*, the remaining birds are all sacrificed correctly as well.

7 Without loss of generality, assume three *olot* from the larger nest, 1 *ḥattat* from the larger nest and 2 *ḥatta'ot* from the smaller nest. Replacing *ḥatta'ot* with *olot* and vice versa, provides an equivalent example.

8 Rambam, *Mishneh Torah*, *Pesulei Ha-Mikdashin* 8:6.

9 The text of the mishnah and its translation is taken from www.Sefaria.com.

general principle: whenever you can divide the pairs [of birds] so that those belonging to one woman need not have part of them [offered] above and part [offered] below, then half of them are valid and half are invalid; but whenever you cannot divide the pairs [of birds] without some of those belonging to one woman being [offered] above and some below, then [the number as there is in] the larger part are valid.

As noted, when birds are sacrificed without consultation, we determine the worst possible case and then determine how many birds are, nonetheless, valid. When *kinnim stumot* are intermingled, the worst possible case invalidates half of each nest, because the worst possible outcome results when every *ken* is sacrificed entirely as either *olot* or *ḥatta'ot*. In many cases, however, more than half of the birds are sacrificed correctly. The second mishnah, conceptually perhaps the hardest mishnah in *Kinnim*, deals with a case when (significantly) **more than half** of the intermingled *kinnim stumot* are valid; in the case in the mishnah, 200 of 232 birds are valid. The difference between what happens with and without consultation can be illustrated in a remarkably simple case when two unspecified nests of unequal size are intermingled. As demonstrated in the introduction, with consultation, only the number of birds in the smaller nest are sacrificed; without consultation, after the fact, the number of birds in the larger nest are valid.

Not so with a *ken mefureshet*, where even when sacrificed without consultation **no birds are valid**; in the worst case, it is possible that every bird designated as an *olah* was sacrificed as a *ḥattat* and every bird designated as a *ḥattat* was sacrificed as an *olah*.

An article by Dr. Philip Reiss[10] provides a thorough formulation, comprehensive explanation, and formal proof of the challenging second mishnah in this chapter. The mishnah specifies that the number of valid sacrifices is the larger amount, *ha-merubeh kasher*. Dr. Reiss formulates this expression precisely, corresponding exactly to the halakhic rule with which the mishnah operates. That rule, as articulated by many classical commentators, is that **the way to derive the minimum number of birds correctly sacrificed is to construct a scenario that maximizes the number of birds incorrectly sacrificed**. This **"smallest majority"** is, minimally, half of the birds; however, under certain scenarios, like that illustrated in the mishnah, it can be considerably larger. In every case, the result is established by constructing the worst case.

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10 Philip Reiss, "A mathematical proof of Kinnim 3:2," *The Torah u-Madda Journal* 9 (2000) 58–75.

When we try to divide the *kinnim* into two groups of equal size and are successful, only half of the birds are valid. When we cannot divide the *kinnim* into two equal sized groups, the number of birds in the larger group is valid. That larger group, however, is defined as the smallest possible larger group. Dr. Reiss formalized what is meant by "the smallest majority" and I will reformulate Dr. Reiss's approach to make the mishnah more intuitive and the proof more concise. It serves as a model for thinking about all the *mishnayot* of the third chapter, which are either examples of this general case or address a situation where *kinnim mefurashot* **and** *kinnim stumot* are intermingled as well. What follows provides a somewhat less formal/rigorous argument that may capture how the mishnah may have been conceived of both by its authors and more traditional commentators.

To formalize the mishnah **and derive the minimum number of birds sacrificed correctly requires construction of a scenario that provably maximizes the number of birds incorrectly sacrificed**. The (minimum number of birds correctly sacrificed) equals

- (the maximum number of birds incorrectly sacrificed)
- subtracted from (the total number of birds).[11]

Without consultation, the Kohen sacrifices half of the combined nest as *ḥatta'ot* and half as *olot*. To develop our model and without loss of generality, assume the Kohen places each of the birds into one of two equal sized storage containers labeled O and Ḥ, which together precisely hold the total number of birds in all of the intermingled nests. The birds in container O are subsequently sacrificed as *olot* while the birds in container Ḥ are sacrificed as *ḥatta'ot*. Since the two containers are of equal size, the Kohen assumes, incorrectly of course, that he has fulfilled his obligation to sacrifice half the nest as *ḥatta'ot* and half as *olot*.

To fully explain and prove the mishnah in its most general case requires that we look at the situation prior to nests being combined and examine the original set of nests. There could have been any number of individual nests, but obviously the number of birds in each of the original nests must be an even number.[12]

Visualize each of the individual *kinnim* with each of its birds stored individually in identical cages; the cages are then stacked vertically one

[11] What is formally demonstrated is that maximizing the number of incorrectly sacrificed birds corresponds to a simple, one-dimensional, container-packing problem. In addition to a formal proof, several examples are provided below.

[12] An even number is expressible as 2*X for some value of X.

on top of another and tied together. That vertical stack of cages from a single *ken* forms a single package. The height of each package is proportional to the (even) number of bird cages in the package; a package holding a *ken* of 4 birds is ½ of the height of a package holding a *ken* of 8 birds.

Since the cages are identical, all nests/packages have the same length and width, which allows them to be stacked (only) vertically in the O and Ḥ containers. Those containers can hold bird cages stacked vertically, but only up to the height of the container.

Return now to the two storage containers of equal size, (the *ḥatta'ot* and *olot* containers respectively,) into which each of the individual cages can be stacked. To match the case in the mishnah, the two identically sized containers are assumed to have the exact capacity required to store all the bird cages in the N packages/*kinnim*, where N denotes the original number of (now intermingled) packages of birds.

Were the birds not intermingled, we can assume that each woman's sacrifice/package is divided in half, with each half temporarily stored in one of the respective containers, before being correctly sacrificed. In the desired scenario, each package (a single *ken*) is split equally across the two containers; the total number of bird cages fit precisely into the two equally sized containers.

To construct a worst-case scenario, we attempt to do exactly the opposite. Instead of dividing each package into an equal number of *ḥatta'ot* and *olot,* we try to leave all the packages unsplit. To the greatest extent possible, something that will be defined formally below, all the birds in each *ken* are placed in **only** one container, i.e., all to be sacrificed as **either** *ḥatta'ot* **or** *olot*. In terms of the two containers, we will try to fill both containers without **splitting** packages, effectively sacrificing all birds from each unsplit package identically, thereby disqualifying exactly half of the birds in that *ken*. When attempting to do that, we may or may not be completely successful. Consider a case of 2, 4 and 6 birds; the worst we can do is put the two smaller *kinnim* with 2 and 4 birds in one container and the larger *ken* of 6 birds in the other. This case illustrates achieving maximal disqualification, where half (6) of the 12 birds are disqualified, but still allows for the valid sacrifice of half (6) of the birds in each nest. However, if we had 2, 4, and 8 birds in each of the original *kinnim*, there is no way to divide the 14 birds into 2 groups of 7, without splitting a package. Note that each *ken* has an even number of birds and the number of birds in any number of unsplit nests will have an even number of birds as well. The worst we can do is to put the two smaller *kinnim* in one container and split the larger *ken* of 8 birds, putting 7 birds in one container and 1 bird in the other. As a result, 2 of the 8 birds in the larger nest are sacrificed correctly, one as an *olah* and one as a *ḥattat*, while half of the remaining 12 birds are sacrificed correctly, for a total of 8, (*ha-merubeh*) correctly sacrificed birds.

If we are trying to place unsplit packages into each container, it should be clear that it is always the case, that at most one package needs to be split across the two containers. This is fundamental to understanding the proof of the mishnah and is proven formally in the footnote below.[13]

There are 2 ways to formalize the mishnah using the paradigm of storing the maximum number of unsplit packages using either **one or both** containers. First, define a 2-container storage scenario as 'optimal' if it maximizes the total number of birds from the unsplit packages that are stored entirely in either of the two containers. The optimization makes the package that must be split as small as possible. Of course, even when the size of one *ken* is not greater than the sum of all of the rest (as in the mishnah), the smallest package that must be split may not be the smallest package.[14] It is not correct to conclude that a scenario that **optimizes the number of unsplit packages/*kinnim*** across both storage containers would **maximize the number of <u>incorrectly</u> sacrificed birds**. Rather,[15] the worst case arises if we try to maximize the number of birds from unsplit packages that can be stored in one single container (as opposed to both.) Instead of trying to fill both containers with unsplit packages, we are trying to place the largest number of unsplit packages in only one of the two containers. What we will prove is that maximizing the number of birds from unsplit packages (in only one container) creates maximal disqualification. A case that illustrates why optimizing one container and not two container storage creates

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13 If a package / *ken* must be split while constructing a worst-case scenario, then one of the containers must be filled completely before any birds / packages are placed in the other container. Were the container not filled, more birds would have to be placed in the other container resulting in more pairs of birds from that *ken* would be sacrificed correctly, something a worst case must avoid. Thus, given that there is no remaining space in one of the containers, no further splitting of any of the packages is possible.

14 In both the case in the mishnah and in the example above of 3 *kinnim* with 2, 4 and 8 birds, one *ken* was larger than the remaining *kinnim*. Consider, however, a case of 6 packages of sizes 6, 20, 28, 28, 28 and 30 where a package of size 20 must be split.

15 An example demonstrating this is given below. It is not easy to construct such examples; I would be interested if anyone can construct a simpler example than the one given.

maximal disqualification and a more intuitive discussion of this issue follows the proof.

If 2*K is the total number of birds in all the *kinnim*, then each container has capacity for K birds.[16] Let J be the largest number of birds that can be placed in one container without splitting a package. Note, if a container (and then, in fact, both containers) can be totally filled without splitting up any package, then J = K. Whether or not the container is full**, the minimal number of correctly sacrificed birds equals J + 2*(K-J).**[17] [18] The formula implies, and that is the essence of what must be proved, that how the remainder of that container and the other container is packed is irrelevant to determining the number of correctly sacrificed birds. (See example d) below.)

Before providing an outline of the proof, some examples will help to illustrate:

a) For 2, 4, 6, 8, and 10 birds (1, 2, 3, 4, and 5 *kinnim*): In this example, K = 15, J = 14, and a minimum of 16 sacrifices are acceptable, since J+2*(K-J) = 14 + 2*(15-14) = 14 + 2 = 16. The first container holds either the groups of 10 and 4 birds or the groups of 6 and 8 birds.

b) For 2, 4, 6, and 8 birds J = K =10: One container holds the groups of 4 and 6 and the other holds the groups of 2 and 8. Since each group of *kinnim* can be stored unsplit, the number of correctly sacrificed bids is exactly half (the worst case.)

c) For the case in the mishnah of 2, 4, 6, 20, 200 birds, K = 116 and J = 32 and at least 200 birds were correctly sacrificed. In this example, the first container holds (2+4+6+20) = 32 birds in unsplit packages and the second container holds **none.** Note that in this case where one of the *kinnim* contains more than half of the total number of birds, this large offering must contain K+ (K-J) birds, the capacity of one container (K) + the remain-

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16 Nothing more than grade-school algebra is required.

17 Note that J + 2*(K-J) equals 2*K – J, the expression for the smallest majority used in Dr. Reiss's paper. As will be explained shortly, 2*K – J is the smallest *merubeh*/majority and results from being able to place J birds from unsplit packages in one container.

18 Thinking of this physically, the remainder of the container is of size (K-J) and comes from a given nest that must have a mate from its nest in the other container, contributing 2*(K-J) valid sacrifices.

ing capacity of the other container (K-J). Trivially, K + (K-J) = J + 2*(K-J).

d) In Dr. Reiss's example of 8, 12, and 14 birds, K= 17 and J = 14 and the minimum number of correctly sacrificed birds = 14 + 2*(17-14) = 20. Note that in the 2-container storage optimization, 14 birds go in one container, 12 in the other while the remaining 8 are split across the two containers – 3 in the container with 14 and 5 in the container with 12. In the *kinnim* case of 1-container optimization however, once 14 birds are placed in one container, how the remaining three slots in that container are filled (i.e. from the group of 8 or 12 or some combination) is irrelevant in determining the number of birds correctly sacrificed. Similarly, in example a) once a container is filled with either 10 and 4 birds, or 6 and 8 birds, which single bird fills the remaining slot in that container is also irrelevant in determining the number of birds correctly sacrificed.

The three types of combined sets of *kinnim* discussed in Dr. Reiss's article are all covered by one algebraic expression. An outline of a formal proof follows.

Remember that J is the largest number of birds that can be placed in one container without splitting any package.[19] Since they are all sacrificed as say *olot*, exactly J/2 were correctly sacrificed. Note that no *ken* has less than (K-J) birds—otherwise, we could have increased the number of birds from unsplit packages in that container. **In fact, (K-J) is less than or equal to half the size of any remaining *ken* since there must be at least (K-J) spaces left in the other container as well. This is the key point in the proof. The other container also holds unsplit packages with less than or equal to J birds and hence there must be room for at least (K-J) additional birds from the remaining package.** Thus, for each of the remaining (K-J) birds placed in that container, both it and some other member of its *ken* were correctly sacrificed (since in all cases at least as many members of <u>**every remaining**</u> *ken* are placed into the other container). This adds 2*(K-J) correctly sacrificed birds. Of the remaining J birds in the second container, <u>**none**</u> have a mate in the first container. Hence, again J/2 are correctly sacrificed. Adding the three groups, the total of correctly sacrificed birds is J/2 + 2*(K-J) + J/2 = J + 2*(K-J) or 2*K-J birds are correctly sacrificed.

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19 Note that J can always be computed, if necessary, by an exhaustive search.

It is critical to appreciate the difference between the improper two container optimization versus the one container optimization that yields the correct number of birds sacrificed correctly after the fact. In examples a) through d) above, both optimizations happen to yield equivalent solutions. However, consider the following sets of offerings: 8, 10, 10, 10, 14, and 14 birds. Note that K = 33 and one package must be split in the 2-container optimization. It is easy to see that the package of size 8 can be split, resulting in one container holding the three packages of size 10 and the other the two packages of size 14. Thus, 58 items in unsplit packages are stored across both containers. If we (incorrectly) set J = 30 corresponding to the number of items from unsplit packages in the larger container, the formula would yield (incorrectly) 36 valid sacrifices. Optimizing only the number of birds from unsplit packages in one container allows you to put 32 = (8 + 10 + 14) items into one container, and hence J = 32. Note that this would not yield an optimization across both containers since only 24 items would go into the second container, with a total of only 56 items stored from unsplit packages across both containers. In this case, the *kinnim* optimization correctly yields only 34 valid sacrifices.[20]

Since constructing that example in 2001, I have tried to find the smallest example. Five *kinnim* of sizes 6, 6, 6, 10 and 10 birds is apparently the smallest example. When maximally filling two containers, we would put *kinnim* with 10 and 6 birds into both containers, leaving one of the three smallest kinnim with 6 birds to be split across both containers. Ostensibly, 22 birds might be assumed correctly sacrificed. But that is incorrect. If we leave all 3 *kinnim* with 6 birds unsplit and placed in one container we get the correct result with only 20 birds correctly sacrificed. We end up splitting a *ken* with 10 birds, 9 going in one container together with the *ken* of 10 birds and the other bird in the same container as the 3 *kinnim* of 6 birds. This demonstrates that 20, not 22, birds is the worst case.[21]

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20 The (8+10+14) birds contribute 16 valid sacrifices. 2 birds (from any other container) are correctly sacrificed. The remaining 32 birds in the second container have no mate in the first container and contribute another 16 valid sacrifices, 34 in total.

21 My nephew, Joshua Blumenkopf, proved that there is no such example with less than 5 *kinnim*. While I assume there is no example with fewer than 38 birds in total across 5 *kinnim*, I do not have a simple proof. One can exhaustively examine 10 through 36 birds.

Conclusions:

I believe this approach renders more intuitive why the maximization of the number of invalid sacrifices requires that one maximize the number of either *olot* or *ḥattot* sacrificed unsplit, into one container (as opposed to both.) Reasoning like that contained in the proof can arguably have been made by *tannaim* 2,000 years ago.

I thank Mel Barenholtz for his comments and an anonymous reviewer for comments that incented me to change the proof in footnote 13.